Lectures in Abelian Varieties
Lecture 1
Complex Tori
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1 Introduction

The moment at which one could say that this area of maths was conceived could be when Riemann submitted his thesis on complex variables. Riemann investigated the theory of compact surfaces and introduced topological methods in complex function theory. Our may interest in Riemanns work is that of the connectivity of srufaces and how this applies to the complex torus.

In these lectures we wish to outline the basic theory of abelian varieties over the complex numbers. An abelian variety is a group variety, which as a variety, is complete. In the classical case, it is not difficult to show that topologically an abelian variety is a **Complex Torus**. We will be following the more analytic side of the subject using both Cornell and Lange et., al. to work through this area.

2 Complex Tori and Abelian Varieties over $\mathbb{C}$

We shall begin with letting V be a complex vector space such that its dimension is g Λ is the lattice in the complex vector space which is by definition a subgroup of rank $2g$ of V, where Λ acts on V in terms of addition. We denote the quotient

$$X = \frac{V}{\Lambda} \tag{2.1}$$

as being a complex torus. Furthermore if we again refer back to the Lange to confirm the fact that the complex torus is a connected complex manifold. So if we assume that X is compact, and since Λ has maximal rank which is described as a discrete subgroup of V and therefore X is the image of the bounded subset of V.

So if we examine the addition action of the lattice as it acts on V, we can see that the induced structure forms an abelian complex Lie group on the torus X. We shall assume that the addition operation behaves in the following manner,

$$\begin{aligned} &\mu : X \times X \to X, \\ &\mu(x_1, x_2) = x_1 + x_2 \end{aligned} \tag{2.2}$$

which forms the addition mapping.

Lemma 2.1. *Any connected compact complex Lie group X of dimension g is a complex torus.*

Proof: If we assume that the torus X is abelian such that the communtator mapping

$$\Phi(x,y) = xyx^{-1}y^{-1} \tag{2.3}$$

where U is the co-ordinate neighborhood for the unit 1 with the torus X. $\forall x \in X$ there exists an open neighborhood denoted V_x of x and with W_x of the unit in X, whereby $\Phi(X_x, W_x) \subseteq U$. Since the commutator mapping with respect to the unit in U is $\Phi(x, 1) = 1$ if and only if Φ is continuous.

As we know that the torus X is compact we can say that there exists a finite number of V_x which cover X. Furthermore if we define W_x to be the intersection of countable open sets W_x. Then we get the following result

$$\Phi(X, W) \subseteq U \implies \Phi(X, W) = 1 \tag{2.4}$$

This is due to the (holomorphism mapping on the compact manifolds are constant) $\Phi(1, x) = 1$ for all $x \in W$. As W is open and non-empty which then implies the original statement that X is abelian.

If we let $\pi : V \to X$ be the universal covering map. The Lie group structure of X induces the simply connected Lie group structure on V, such that π is a homomorphism. We can assume say that as X abelian so to by definition is V. Therefore V is isomorphic to the vector space $\mathbb{C}^g$. The compactness of X implies that the $ker(\pi)$ is a lattice in V. Finally we can say that the lie group X of dim g is a complex torus.

The vector space V which we defined as the universal covering space, may also be considered as the universal covering map. The kernel Λ of π can be identified with the fundamental group $\pi_1(X) = \pi_1(X, 0)$. As Λ is abelian, $\pi_1(X)$ is canonically isomorphic to the first homology group $H_1(X, \mathbb{Z})$.

We should also know that the tangent space $T_0 X of X$ in 0 is given by the fact that the torus is locally isomorphic to V. Furthermore we can see that the mapping $\pi : V = T_0 X \to X$ is simply an exponential map.

We should now consider an example of the case where $g = 1$. This is the case of the Elliptic curve, where a 1-dimensional complex torus describes a basis which may admit V with the field of complex numbers $\mathbb{C}$. The lattice Λ within $\mathbb{C}$ is generated by 2 complex numbers λ_1 and λ_2 which are linearly independent over the real numbers. Without drawing the parallelogram we should just imagine a clean sheet of A4 paper. Simply roll the paper into a cylinder and join either end together. If we think of this transformation in terms of this physical change from sheet to a donut we can visualize the mapping $\pi : V \to X$. To capture this concept in ones mind will allow us to generalize this result.

The general case enables us to describe the complex torus in a more formal fashion. To describe the torus we choose a bases $e_1, \ldots, e_g$ of V, and $\lambda_1, \ldots, \lambda_{2g}$ of the lattice Λ. We can also write λ_i in terms of the basis $e_1, \ldots, e_g : \lambda_i = \sum_{j=1}^{g} \lambda_{ji} e_j$. The matrix

$$\prod = \begin{pmatrix} \lambda_{11} & \cdots & \cdots & \lambda_{1,2g} \\ \vdots & & & \vdots \\ \lambda_{g1} & \cdots & \cdots & \lambda_{g,2g} \end{pmatrix} \tag{2.5}$$

in $M(g \times 2g, \mathbb{C})$ is called a period matrix for X. The period matrix $\prod$ determines the complex torus X. However we should restate the important fact that one must choose the bases for V and Λ.

Proposition 2.2. $\prod \in M(g \times 2g, \mathbb{C})$, *is the period matrix of a complex torus if and only if the matrix* $P = \binom{\prod}{\overline{\prod}} \in M_{2g}(\mathbb{C})$ *is nonsingular, where* $\overline{\prod}$ *denotes the complex conjugate matrix*

Proof: $\prod$ is a period matrix if and only if the column vectors of $\prod$ span a lattice in $\mathbb{C}^g$, which implies that the columns are linearly independent over the real numbers.
If we assume that the columns of $\prod$ are linearly dependent over $\mathbb{R}$. Then there is an $x \in \mathbb{R}^{2g}, x \neq 0$, with $\prod x = 0$, and we get $Px = 0$. This implies $detP = 0$.

Conversely, if P is singular, there are vectors $x, y \in \mathbb{R}^{2g}$, not both zero, such that $P(x+iy) = 0$. But $\prod(x+iy) = 0$ and $\prod(x - iy) = \overline{\overline{\prod}(x+iy)} = 0$ imply $\prod x = \prod y = 0$. Hence the columns of $\prod$ are linearly dependent over $\mathbb{R}$.

So having described the general form of the complex torus both in general and in terms of the manifold and vector space definitions we will move on to the next lecture.