Hermitian Structures in Braided Pre-monoidal Categories

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Abstract

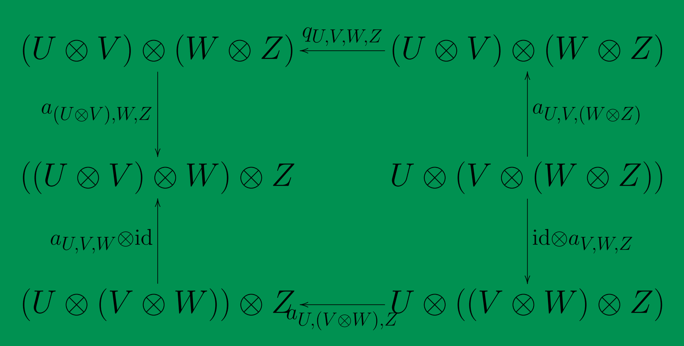
Lie group symmetries are well known to play a role in the physics of quantum systems. The representation theory of Lie groups can be conveniently expressed in the framework of category theory. Category theory also extends to incorparate particle statistics, which in turn allows for an investigation of exclusion and confinement principles through the action of the symmetric groups. We go on to investigate the hermitian structure over a symmetric premonoidal category of representations over a Lie group. We will examine this general approach in terms of categorical traces and hermitian forms.

1 Braided Pre-monoidal Categories

Pre-monoidal categories are used within the Racah-Wigner calculus to reformulate the physical context and interpretation of Feynman diagrams [5]. This work lead to the development of examining the boson-fermion statistic for $su(3)$ colour [6]. The only possible construction is to form a strictly pre-monoidal category which deforms the pentagon condition. We shall begin this section by defining a pre-monoidal category, taken from [3].

Definition 1. *A pre-monoidal category is a triple $(\mathcal{C}, \otimes, \mathcal{A})$ where $\mathcal{C}$ is the class of objects, $\otimes$ is the bifunctor $\otimes : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ and A is the natural associator isomorphism such that $A : \otimes(id \times \otimes) \to \otimes(\otimes \times id)$.*

We define the associator A, as being the natural associator. However in practice we use $a_{U,V,W}$ as a member of A for all objects $U, V, W \in \mathcal{C}$. Therefore $a_{U,V,W}$ is the same morphism which we have described previously. We should note however that there are no conditions imposed on A, but it is important that we should define the natural isomorphism $Q : \otimes(\otimes \times \otimes) \to \otimes(\otimes \times \otimes)$ via the following diagram:



This box diagram expresses Q in terms of the component isomorphism which are defined by [3] as:

$$Q((U \otimes V) \otimes (W \otimes Z)) = q_{U,V,W,Z}((U \otimes V) \otimes (W \otimes Z)) \tag{1}$$

which may be rewritten as,

$$q_{U,V,W,Z} = a^{-1}_{(U\otimes V),W,Z}(a_{U,V,W} \otimes id)a_{U,(V\otimes W),Z}(id \otimes a_{V,W,Z})a^{-1}_{U,V,(W\otimes Z)}. \tag{2}$$

This deformed pentagon condition is simply a more generalised form which we used in describing monoidal categories. We must examine the significance of Q and its use in distinguishing the coupling of the objects of the category through the brackets $[]$, $\{\}$. This notation is shown by [3],

$$Q([U \otimes V] \otimes \{W \otimes Z\}) = (\{U \otimes V\} \otimes [W \otimes Z]) \tag{3}$$

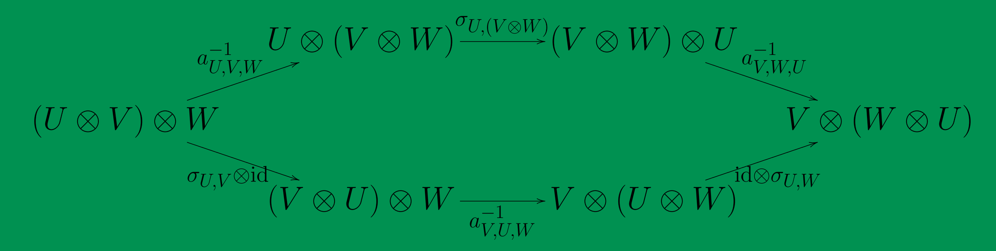
as being the functor Q which provides the temporal coupling of U and V before coupling W and Z as being distinguishable from the reverse coupling.

We can describe pre-monoidal categories as being unital if they have an identity object $\mathbf{1} \in \mathbb{C}$ and natural isomorphisms $\rho_U : U \otimes \mathbf{1} \to U$ and $\lambda_U : \mathbf{1} \otimes U \to U$.

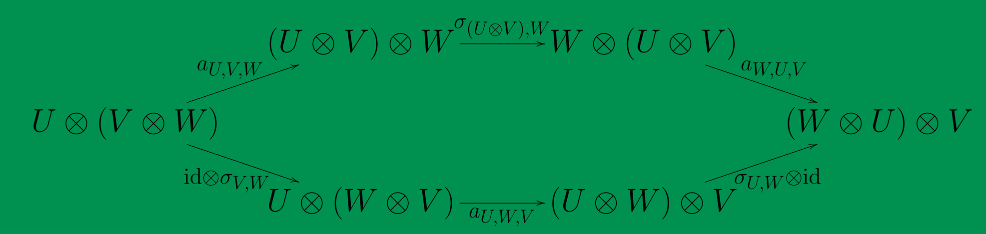
The next important class of unital pre-monoidal categories is when the tensor product is commutative up to isomorphism. Ths leads to the concept of a braided unital pre-monoidal category.

Definition 2. *A unital pre-monoidal category $\mathbb{C}$ is said to be braided if it is equipped with a natural commutativity isomorphism $\sigma_{U,V} : U \otimes V \to V \otimes U$ for all objects $U, V \in \mathbb{C}$ such that the following diagrams commute:*

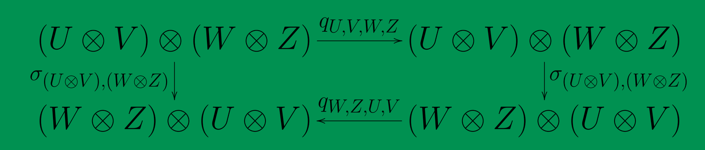
(i)



(ii)



(iii)



We should also make clear to the reader that a unital, pre-monoidal category for which

$$Q = id \otimes id \otimes id \otimes id \tag{4}$$

is a monoidal category. Furthermore we must consider the principle of rigidity. Rigidity is related to the inclusion of dual objects in the pre-monoidal category in a consistent way. All of the diagrams which were outlined for rigidity above apply to pre-monoidal categories. Thus we can say that a unital pre-monoidal category is rigid if every object has a dual object and if the dual object functor is an (anti-)equivalence of categories.

2 Twining

We know how to use quasi-triangular quasi-bialgebras to construct braided monoidal categories [2]. However we need to use a slight different method to build a pre-monoidal category. We shall use the work of [3] to show how to construct such a category.

If we define A as a quasi-triangular quasi-bialgebra by the pentuple $(A, \Delta, \varepsilon, \tilde{\Phi}, \tilde{\mathcal{R}})$ with a Casimir invariant K and a fixed but arbitary $\gamma \in \mathbb{C}$. The following relations also hold;

$$\begin{aligned}\tilde{\mathcal{R}} &= \gamma^{K\otimes K}\mathcal{R} = \mathcal{R} \cdot \gamma^{K\otimes K} \\ \tilde{\Phi} &= \Phi \cdot \gamma^{\kappa}\end{aligned} \tag{5}$$

such that $\kappa = K \otimes (I \otimes K + K \otimes I - \Delta(K))$. Furthermore we should also note that the action of $\tilde{\Phi}$ is slightly different to that of Φ as shown in equation (2) of [3]. Thus the following must hold;

$$\begin{aligned}&(id \otimes \Delta)\Delta(a)\tilde{\Phi}^{-1}(\Delta \otimes id)\Delta(a)\tilde{\Phi} \ \ \forall a \in A \\ &\tilde{\mathcal{R}}\Delta(a) = \Delta^T(a)\tilde{\mathcal{R}} \\ &(\Delta \otimes id)\tilde{\mathcal{R}} = \tilde{\Phi}_{312}\tilde{\mathcal{R}}_{13}\tilde{\Phi}_{132}\tilde{\mathcal{R}}_{23}\tilde{\Phi}^{-1}_{123} \\ &(id \otimes \Delta)\tilde{\mathcal{R}} = \tilde{\Phi}^{-1}_{213}\tilde{\mathcal{R}}_{13}\tilde{\Phi}^{-1}_{213}\tilde{\mathcal{R}}_{12}\tilde{\Phi}_{123}(\gamma^{2K})^{-1}_{123}\end{aligned} \tag{6}$$

Isaac et.al. [3] goes on to define

$$\xi = (\Delta \otimes id \otimes id)\tilde{\Phi}^{-1}(\tilde{\Phi} \otimes I) \cdot (id \otimes \Delta \otimes id)\tilde{\Phi} \cdot (I \otimes \tilde{\phi}) \cdot (id \otimes id \otimes \Delta)\tilde{\Phi}^{-1} \tag{7}$$

and notes that the quasi-Yang-Baxter equation is satisfied. These relations show that the catgeory of A-modules with

$$a_{U,V,W} = (\pi_U \otimes \pi_V \otimes \pi_W)\tilde{\Phi}, \tag{8}$$

such that $mod_K(A)$ is a premonoidal category as $\tilde{\Phi}$ fails the pentagon condition. Thus the representation

$$q_{U,V,W,Z} = (\pi_U \otimes \pi_V \otimes \pi_Z)\xi \tag{9}$$

does not act as the identity. However, $\tilde{\mathcal{R}}$ cannot be used to construct a braided, pre-monoidal category of A-modules, due to the term $\gamma^{2\kappa}$ in (5) which violates the hexagon condition (i) of Definition 2. On the other hand, it may well be possible that for suitably chosen γ there is a subcategory of $\mathbf{mod}_K(A)$ for which $\gamma^{2\kappa} = 1$ when restricted to this subcategory. In this case, such a subcategory may acquire the structure of a rigid, braided, pre-monoidal category. Below we demonstrate that this is indeed possible when A is the universal enveloping algebra $U(g)$ of a simple Lie algebra g. Moreover, we will see that the subcategory is the full category $\mathbf{mod}_K(A)$ containing all the finite-dimensional irreducible $U(g)$-modules. The universal enveloping algebra $U(g)$ of a Lie algebra g acquires the structure of a quasi-bialgebra with the mappings

$$\begin{aligned}&\epsilon(I) = 1, \ \epsilon(x) = 0, \ \forall x \in g \\ &S(I) = I, \ S(x) = -x, \ \forall x \in g \\ &\Delta(I) = I \otimes I, \ \Delta(x) = I \otimes x + x \otimes I, \ \forall x \in g\end{aligned} \tag{10}$$

which are extended to all of $U(g)$ such that ϵ and Δ are algebra homomorphisms and S is an anti-automorphism. It is easily checked that Δ is co-associative; i.e. $(\mathrm{id} \otimes \Delta)\Delta(x) = (\Delta \otimes \mathrm{id})\Delta(x) \ \forall x \in U(g)$. This means that we can take $\Phi = I \otimes I \otimes I$ for the co-associator of $U(g)$ and $\alpha = \beta = I$. We must also set $\epsilon(K) = 0$ and assume that $S(K) = -K$. In this work we shall adopt the summation convention over all repeated indicies such that

$$\begin{aligned}\Phi &= X_j \otimes Y_J \otimes Z_j \\ \Phi^{-1} &= \bar{X}_j \otimes \bar{Y}_J \otimes \bar{Z}_j \\ \mathcal{R} &= \sum_i a_i \otimes b_i\end{aligned} \tag{11}$$

The u operator is the next major calculation which needs to be shown here. By proving that the u operator is unique and non-trivial we can apply Altschuler's [1] theory that for any finite dimensional representation of A the double dual is equivalent to the original representation. Thus the left and right duals are equivalent, which leads to the examination the observable states of confinement. We shall now calculate the u operator:

$$\begin{aligned}\bar{X}_j S(\bar{Y}_j)\bar{Z}_j &= (-1)^{-K(K-K-\epsilon(K))} \\ &= I.\end{aligned} \tag{12}$$

$$\begin{aligned}S(X_j)Y_j S(Z_j) &= (-1)^{-K(K-K-\epsilon(K))} \\ &= I.\end{aligned} \tag{13}$$

$$S(b_l)a_l = (-1)^{-K^2}. \tag{14}$$

$$\begin{aligned}u &= S(Y_j S(Z_j))S(b_l)a_l X_j \\ &= S(Y_j S(Z_j))X_j S(b_l)a_l \\ &= S(Y_j S(Z_j))X_j(-1)^{-K^2}. \\ &= (-1)^{-K^2}\end{aligned} \tag{15}$$

3 Hermitian Structures

In this investigation we must explore the notion of a hermitian category. We owe all the definitions and properties used here to Turaev [8] and Kirillov [7]. We shall continue to assume that the ground field used here is $\mathbb{C}$.

We shall use a ribbon category $\mathcal{V}$ which is defined over $\mathbb{C}$ with an involution mapping $\bar{f} : V \to U$ such that $f : U \to V$, via complex conjugation. Furthermore the involution morphism, is such that the following identities hold:

$$\bar{\bar{f}} = f, \ \overline{f \otimes g}, \ \bar{f} \otimes \bar{g}, \ \overline{f \circ g} = \bar{g} \circ \bar{f}, \tag{16}$$

such that f and g are arbitrary morphism in and $f \circ g$ is an arbitrary compossible morphism both of which are in $\mathcal{V}$. For any $Ob(U) \in \mathcal{V}$ we get the following trail of identities:

$$i\bar{d}_U = i\bar{d}_U id_U = i\bar{d}_U i\bar{\bar{d}}_U = \overline{i\bar{d}_U id_U} = i\bar{\bar{d}}_U = id_U \tag{17}$$

The hermitian ribbon category is endowed with $\sigma_{\bar{U},V} = \sigma^{-1}_{U,V}$ for any object U of $\mathcal{V}$ we get,

$$\begin{aligned}&\bar{c}_U = (c_U)^{-1} \\ &e\bar{v}_U = coev_U \sigma_{U,U^*}(c_U \otimes id_{U^*}) : U^* \otimes U^{**} \to 1 \\ &co\bar{e}v_U = (id_{U^*} \otimes c_U^{-1})\sigma^{-1}_{U^*,U} ev_U : 1 \to U^* \otimes U^{**}\end{aligned} \tag{18}$$

So if $\mathcal{V}$ is a hermitian monoidal category we can easily see that from a geometric interpretation of the bar conjugtion we can get the following identites from Kirillov [7], which show that $d\bar{i}m\ U = dim\ U$

$$\begin{aligned}\bar{s_{ij}} &= s_{ij^*} \\ \bar{c}_i &= c_i^{-1}.\end{aligned} \tag{19}$$

Our work will move on to describing how to:

- Give a correct description for a hermitian structure [8] over a symmetric, pre-monoidal category of representations for any Lie group.
- Show association of hermitian structure with the notion of observability of the states of the system.
- Explore notion that the space of confined states, cannot be consistently endowed with an sesquilinear form invariant with respect to the Lie group action, needed to define a hermitian structure over the category of representations.
- Define, for an arbitrary Lie group G and associated symmetric, pre-monoidal category of representations prescribed by a Casimir element K, the algebra of observables $A(G, K)$.
- In general, $A(G, K)$ will be a non-linear generalisation of a Lie superalgebra along the lines recently proposed in [4].
- The definition for $A(G, K)$ in the present context will be required to account for the non-associativity. As such, new classes of algebraic structures will be uncovered.

References

[1] Altschuler, D., Coste, A.: Quasi-quantum groups, knots, three-manifolds and topological field theory. Commun. Math. Phys. **150** (1992) 83-107

[2] Chari, V.C., Pressley, A.: *A Guide to Quantum Groups* (Cambridge University Press, 1994)

[3] Isaac, P., Joyce, W.P., Links, J., *An Algebraic Origin for Quark Confinement* arXiv:hep-th/0307046 v1 4 July 2003"

[4] Jarvis, P.D., Rudolph, G., *Polynomial super-gl(n) algebras* J. Phys. A: Math. Gen. **36**(2003) 5531-5555

[5] Joyce, W.P., *Vertex Calculus in Premonoidal Categories: I. The Racah-Wigner Calculus, Recoupling and Statistics* Preprint 2002 (Submitted to J. Math. Phys.)

[6] Joyce, W.P., *Quark Confinement as a Consequence of the Extension of the Boson/Fermion Statistic to $SU(3)$ Colour* J. Phys A vol.36 (2003) no.49 pp.12329-12431

[7] Kirillov, A.A., *On Inner Product In Modular Tensor Categories* 1996, Journal of the American Mathematical Society, vol.9, no.4 pp.155-180

[8] Turaev, V., *Quantum invariants of knots and 3-manifolds*, (W. de Gruyter, Berlin, 1994).