Lectures in Abelian Varieties
Lecture 3
Line Bundles and Abelian Varieties
by
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1 Introduction

In this lecture we shall endevour to enable to reader to understand the basic concepts behind line bundles. We will adopt the approach of Lange, by defining a holomorphic line bundle on a complex torus as the factors of automorphy.

2 Line Bundles

So if we say that the complex manifold X is the complex torus or the universal abelian variety over the Siegel upper half space. Then if we let π:$\tilde{\mathrm{X}}\rightarrow X$ be the universal covering. The topological space $\tilde{\mathrm{X}}$ derives the structure of the universal covering structure of a complex manifold. We use Lange's notation whereby $\pi_1(X)$ as the fundamental group of π :$\tilde{\mathrm{X}}\rightarrow X$. We should further note that by convention the base point is omitted.

So we should now aim to describe the holomorphic map,

$$f : \pi_1(X) \times X \rightarrow \mathbb{C}^* \tag{2.1}$$

such that we obtain the cocycle relation

$$f(\lambda\mu, x) = f(\lambda, \mu) f(\mu, x) \tag{2.2}$$

$\forall\ \lambda, \mu \in \pi_1(X),\ x \in X$ is the 1-cocycle of $\pi_1(X)$ with $H^0(\mathcal{O}_X^*)$. If we examine the 1-cocycles we can see that they form an abelian group $Z(\pi_1(X), H^0(\mathcal{O}_X^*))$. We refer to the factors of this group as the factors of automorphy. These factors form the mapping

$$(\lambda, x) \mapsto h(\lambda x) h(x)^{-1} \tag{2.3}$$

for some $h \in H^0(\mathcal{O}_X^*)$ are boundaries.

So the subgroup $B'(\pi_1(X), H^0(\mathcal{O}_X^*))$ of $Z(\pi_1(H), H^0(\mathcal{O}_X^*))$. The group

$$H'(\pi_1(X), H^0(\mathcal{O}_X^*)) = \frac{Z'(\pi_1(X), H^0(\mathcal{O}_X^*))}{B'(\pi_1(X), H^0(\mathcal{O}_X^*))} \tag{2.4}$$

Any f in $Z'(\pi(X), H^0(\mathcal{O}_X^*))$. This defines the **Line Bundle** on X as follows:

Lemma 2.1. *If we consider the holomorphic action of $\pi_1(X)$ on a trivial bundle $X \times \mathbb{C} \to X$ we get*

$$\lambda \circ (x, t) = (\lambda x, f(\lambda, xt)) \tag{2.5}$$

$\forall \lambda \in \pi_1(X)$.

So the action is free and properly discontinous, so the quotient $L = X \times \mathbb{C}/\pi_{1(X)}$ is a complex manifold. If we consider the projection $p : L \to X$ induced by $X \times \mathbb{C} \to X$ one can easily see that L is a holomorphic line bundle on X.

So if we examine both Abe et.al., and Lange then we can see by definition the group of holomorphic line bundles on X is canonically isomorphic to the group $H'(\mathcal{O}_X^*)$. Therefore we can show that this map induces an injective homomorphism of cohomology groups.

Proposition 2.2.

$$\phi_1 : H'(\pi_1(X), H^0(\mathcal{O}_X^*) \to ker\ (H'(X, \mathcal{O}_X^*)) \overset{\pi^*}{\to} (X, \mathcal{O}_X^*) \tag{2.6}$$

Remarks: We will simply refer to Lange for this proof to avvoid 15 pages or so of unnessecary pain.

So having intoriduced the concept of Line Bundles we shall now work through the first major lemma of Lange's book. The first Chern class $c_1(L)$ for any holomorphic line bundle L on in terms of a factor of autmorphy of L. This leeds us to trying to determin the Néron-Severi group of X.

We shall use the Picard group Pic(X) to denote the group of line bundles on X. This can also be described as $H^1(X, \mathcal{O}_X^0)$ naturally. As we have showed above we can describe a line bundle on X whose pullback to V is trivial, can be described by its factor of automorphy. We shall now assert using the following lemma that this is true for every line bundle on X.

Lemma 2.3. *Every holomorphic line bundle on a complex vector space V is trivial*

Proof: If we use the natural progress from the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_V \overset{e(2\pi i)}{\to} \mathcal{O}_V^* \to 1 \tag{2.7}$$

we get the exact sequence

$$H^1(\mathcal{O}_V) \to H^1(\mathcal{O}_V^*) \to H^2(V, \mathbb{Z}). \tag{2.8}$$

But $H^1(\mathcal{O}_V) = 0$ by the Poincare Lemma. However from Rotman we know that $H^2(V, \mathbb{Z}) = 0$. Using this we can imply our assertion.

Furthermore any holomorphic line bundle on X can be described by a factor of autmorphy. Thus using the first Chern class of a line bundle we use the exponential sequence and its long cohomology sequence

$$\longrightarrow H^1(X,\mathbb{Z}) \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow \tag{2.9}$$

We describe the image $c_1(L)$ of a line bundle $L \in H^1(\mathcal{O})$ in $H^2(X,\mathbb{Z})$ is called the first Chern class of L. According to Lange the groups $H^2(X,\mathbb{Z})$ and $Alt^2(\Lambda,\mathbb{Z})$ are canonically isomorphic. Therefore we consider $c_1(L)$ as an alternating $\mathbb{Z}$-valued form on the lattice Λ.