Lectures in Abelian Varieties
Lecture 2
Homomorphisms
by
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1 Homomorphisms

We shall now describe the two types of holomorphic maps between complex tori which are translations and homomorphisms. We shall see that every holomorphic mapping is composed of one of each.

So if we consider two tori such that $X = V/\Lambda$ and $X' = V'/\Lambda'$ with dimensions g and g' respectively. The homomorphism of $h : X \to X'$ is a holomorphic map, compatible with the group structures. Furthermore the translation by an element $x_0 \in X$ is defined to be the holomorphic map such that $t_{x_0} : X \to X,\ x \mapsto x + x_0$.

Proposition 1.1. *Let $h : X \to X'$ be a holomorphic function map.*

1. *There is a unique homomorpism $h : X \to X'$ such that $h = t_{h(0)}f$, i.e. $h(x) = f(x) + h(0)$ for all $x \in X$*
2. *There is a unique $\mathbb{C}$-linear map $F : V \to V'$ with $F(\Lambda) \subset \Lambda'$ inducing the homomorphism f.*

Proof Define $f = t_{-h(0)}h$. We can lift the composed map $\pi : V \to X$ and $f : X \to X'$ to a holomorphic map F into the universal covering V' of X', such that

$$\begin{aligned} F :& V \to V' \\ f\pi :& V \to X' \\ \pi' :& V' \to X' \end{aligned} \tag{1.1}$$

in such a way that $F(0) = 0$. The diagram implies that for all $\lambda \in \Lambda$ and $v \in V$ we have $F(v + \lambda) - F(v) \in \Lambda'$. Thus the continous map $v \mapsto F(v + \lambda) - F(v)$ is constant and we get $F(v + \lambda) = F(v) + F(\lambda)$ for all $\lambda \in \Lambda$ and $v \in V$. Hence the partial derivatives of F are 2g-fold periodic and thus constant by Liouville's theorem. It follows that F is $\mathbb{C}-$linear and f is a homomorphism. The uniqueness of F and f is obvious.

If we consider the set of homomorhpisms under the addition mapping of X into X', then we can form an abelian group which is denoted as $Hom(X, X')$. We should know that the holomorphic map from above gives us an injective homomorphism of abelian group

$$\rho_a : Hom(X, X') \to Hom_{\mathbb{C}}(V, V'), f \mapsto F \tag{1.2}$$

By achknowledgeing the restriction for F_Λ on F with respect to the lattice Λ as being $\mathbb{Z}-$linear we can see that F_Λ determines F and f completely. Therefore we can construct the injective homomorphism such that

$$\rho_r : Hom(X, X') \to Hom_{\mathbb{C}}(V, V'),\ f \mapsto F_\Lambda \tag{1.3}$$

This rational representation of $Hom(X, X')$, with the extension of ρ_a and ρ_r to $Hom_{\mathbb{Q}}(X, X') = Hom(X, X') \otimes_{\mathbb{Z}} \mathbb{Q}$ by the same expression. Furhermore we should refer to these morphism as being analytic and rational representations. As any subgroup pf $Hom_{\mathbb{Z}}(\Lambda, \Lambda') \cong \mathbb{Z}^{4gg'}$ is isomorphic to $\mathbb{Z}^m$, with injectivity being implied by ρ_r.

Proposition 1.2. *$Hom(X, X') \cong \mathbb{Z}^m$ for some $m \leq 4gg'$.*

If we look further and describe another injective homomorphism such that $X" = V"/\Lambda"$ is another complex torus and $f \in Hom(X, X')$ and $f' \in Hom(X', X")$, therefore we get

$$\rho_a(f'f) = \rho_a(f')\rho_a(f) \tag{1.4}$$

(the natural distribution map).

Thus from the previous proposition we can use the uniqueness of F and f. Furthermore if $X = X'$, ρ_a and ρ_r are representations of the endomorphism ring, $End(X)$ and $End_{\mathbb{Q}} := End(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. If we consider the two period matrices such that $\Pi \in M(g \times -2g, \mathbb{C})$ and $Pi' \in M(g' \times 2g', \mathbb{C}$ for X and X' with respect to V, Λ and V', Λ'. Clearly then $f : X \to X'$ is a homomorphism. Now the choosen bases of $\rho_a(f)$ and $\rho_r(f)$ for $A \in M(g' \times g, \mathbb{C})$ and $R \in M(2g' \times g, \mathbb{Z})$ respectively.

The Matrix condition $\rho_a(f)(\Lambda) \subset \Lambda'$ such that

$$A\Pi = \Pi' R \tag{1.5}$$

Any 2 matrices A and R (as defined before) that satisy 1.5 define the homomorphism $X \to X'$. Therefore ρ_a and ρ_r are related. This brings us to the following proposition.

Proposition 1.3. *The extended ratioanl representation*

$$\rho_r \otimes 1 : End_{\mathbb{Q}}(X) \otimes \mathbb{C} \to End_{\mathbb{C}}(\Lambda \otimes \mathbb{C}) = End_{\mathbb{C}}(V \times V) \tag{1.6}$$

is equivalent to the direct sum of the analytic representation and its complex conjugate: $\rho_r \otimes 1 \cong \rho_a \oplus \overline{\rho}_a$

We shall not prove this statement as the description above outlines the details needed. As a natural conseqence we will now examine the image and kernal of the homomorphism $f : X \to X'$ of the complex tori.

Proposition 1.4.

1. *imf is a subtorus of X'*
2. *$kerf$ is a closed subgroup of X. The connected component $(kerf)_0$ of $kerf$ containing 0 is a subtorus of X of finite index in $kerf$.*

This appears as the conseqence of lemma 1.1 from the previous lecture. Thus the commutator complex lie group X of dim g is a complex torus. Having said this we should work through an example of this statement to demonstarte our assumption of the proof. So $X \times X'$ of complex tori is the complex torus $X \times X' = V \times V'/\Lambda \times \Lambda'$. The projection of $X \times X'$ onto its factors and natural embedding with respect to $X \times X'$ is a homomorphism of complex tori. The same can also be said for X' with respect to $X \times X'$ is aslso a homomorphism of complex tori. Furthermore the the analytic representation of X is just the projections natural embeddings of the corrosponding vector space. This leads us to the further statement for X' such that the rational representation of this homomorphism is a projection and natural embedding of the lattice Λ.

1.1 Isogenies

We now need to define a specfic class of homomorphism of complex tori. This is simply the surjection homomorphism $X \to X'$ with a finite kernel. A homomorphism is an isogeny if and only if is is a surjection mapping with $dim\ X = dim\ X'$. So if $\Gamma \subseteq X$ is a finite subgroup then X/Γ defines a complex torus with the natural projection

$$p : X \to X/\Gamma \tag{1.7}$$

is therefore an isogeny. We should also be able to see the fact that $\pi^{-1}(\Gamma) \subset V$ is a lattice containing Λ and $X/\Gamma = V/\pi^{-1}(\Gamma)$ and thus up to the isomorphism every isogeny is of this form. By the above proposition 1.4 we can see that every surjective homomorphism $f : X \to X'$ of complex tori which factors into a canonical form with respect to a surjective homomorphism g. Having a complex kernel and an isogeny h we are faced with the Stein factorization of f,

$$\begin{aligned} f :& X \to X' \\ g :& X \to X/(kerf)_0 \\ n :& X/(kerf)_0 \to X' \end{aligned} \tag{1.8}$$

The degree (deg f) of a homomorphism $f : X \to X'$ to be $ord(ker\ f)$ if it is finite and 0 otherwise. Thus for any isogeny we get

$$deg\ f = (\Lambda' :\ deg \varrho_r(f)(\Lambda)) \tag{1.9}$$

and its index is $\varrho_r(f)(\Lambda)$ in Λ'. If f is the endomorphism $End(X)$ and $\Lambda = \Lambda' \therefore det\ f = det\ \varrho_r(f)$. Further more we should note that $det \varrho_r(f)$is positive by proposition 1.3 and the above statement is vaild for an arbitrary endomorphism as both sides are zero if f is not an isogeny. Therefore if we take f as being surjective and $g : X' \to X$" is the 2nd homomorphism. Then we have for the degree for the composition $deg\ gf = deg\ f \cdot deg\ g$. thus if f and g are isogenies, then the composition of gf is therefore by definition an isogeny. For any integer n we define the homomorphism $n_X : X \to X$ by $x \mapsto nx$. If $n \neq 0$, its kernel X_n is called the group of n-division points of X.

Proposition 1.5. $X_n \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$

Proof $ker\ n_X = \frac{1}{n}\Lambda/\Lambda \cong \Lambda/n\Lambda \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$

This statement implies that for any $n \neq 0$ that the homomorphism n_X is an isogeny of degree n^{2g}. Furthermore any complex torus is a divisble group and $Hom(X, X')$ is a torsion free as an abelian group. Thus $hom(X, X')$ should be considered as the subgroup of $Hom_{\mathbb{Q}}(X, X')$ by

$$deg(rf) := r^{2g} deg\ f \tag{1.10}$$

for any $r \in \mathbb{Q}$ and $f \in Hom(X, X')$. We will see that isogenies are almost but not quite a isomorphism. We define the exponent $e = e(f)$ of an isogeny f to be the exponent of the finite group $ker\ f$. In other words $e(f)$ is the smallest positive integer n with $nx = 0$ for all x in $ker\ f$.

Proposition 1.6. *For any isogeny $f : X \to X'$ of exponent e there exists an isogeny $g : X' \to X$, unique up to isomorphisms, such that $gf = e_X$ and $fg = e_{X'}$*

Proof: As $Ker\ f \subseteq ker\ e_X$, there is a map $g : X' \to X$ such that $gf = e_X$ With e_X and f also g is an isogeny. The kernel of g is continued in the kernel of $e_{X'}$, since for every $x' \in ker\ g$ there is an $x \in kere_X$ with $f(x) = x'$ and $ex' = ef(x) = f(ex) = 0$. Thus $e_{X'} = f'g$ for some isogeny $f' : X \to X'$ and we get $f'e_X = f'gf = e_{X'}f = fe_X$. This implies $f = f'$, since e_X is surjective.

Corollary 1.7. *1. Isogenies define an equivalence relation on the set of complex tori.*

2. An element in End(X) is an isogeny if and only if it is invertible in $End_{\mathbb{Q}}(X)$

Therefore we can clearly call two complex tori isogenous, if there is an isogeny between them.