Lectures in Abelian Varieties
Lecture 0
Background
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In this brief lecture we will simply give a short outline of the fundamental definitions and background required to undertake a tour of abelian varieties. Here we shall just review a few things from third year algebra and complex analysis to refresh our working knowledge.

Definition 0.1. ***Holomorphic Function*** *If the derivative $f'(z)$ exists at all points z of a region $\mathcal{R}$, then $f(z)$ is said to be analytic in $\mathcal{R}$ and is referred to as an analytic function in $\mathcal{R}$ or a function in $\mathcal{R}$ or a function analytic in $\mathcal{R}$. The terms regular and Holomorphic are often used in algebra texts.*

Furthermore a function $f(z)$ is said to be analytic at a point z_0 if there exists a neighbourhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

Definition 0.2. ***Meromorphic Functions*** *A function which is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function.*

Definition 0.3. *A **Semi-Lattice** is a semi-group $\langle A, \wedge \rangle$ with properties*

$$\begin{aligned} &a \wedge b = b \wedge a \\ &\quad \textit{and} \\ &a \wedge a = a \\ &\forall a, b \in A \end{aligned} \tag{0.1}$$

A typical example of a semilattice is formed by taking A to be a collection of all subsets of an arbitrary set with the operation being intersection.

Definition 0.4. ***Lattices*** *A lattice is an algebra, $\langle A, \wedge, \vee \rangle$ such that both $\langle A, \wedge \rangle$ and $\langle A, \vee \rangle$ are semi-lattices and the following two equalities hold:*

$$\begin{aligned} &a \vee (a \wedge b) = a \\ \textit{and}& \\ &a \wedge (a \vee b) = a. \\ &\forall a, b \in A \end{aligned} \tag{0.2}$$

An example of a lattice is based upon A being the collection of all equivalence relations on an arbitrary set, $\wedge$ which is the intersection operation and $\vee$ to be transitive closure of the union of two equivalence relations.

We should also know the fact that a lattice in a complex vector space $\mathbb{C}^g$ is by definition a discrete subgroup of maximal rank in $\mathbb{C}^g$ which is also a free abelian group of rank $2g$. As we will see later this is of great significance in terms of the complex torus the variety is mapped over.

Corollary 0.5. *Let X be a complex manifold and suppose G is a group acting freely and properly discontinuously on X. Thus X/G is also a complex manifold.*

Definition 0.6. ***Lie Group*** *Let G be a smooth manifold which is also a topological group with multiplication map $mult : G \times G \to G$ and the inverse map $inv : G \to G$ and view $G \times G$ as the product manifold. Then G is a Lie Group if mult and inv, are smooth maps.*

Definition 0.7. ***Lie Algebra*** *An algebra L equipped with a multiplication map $[\ ,\ ] : L \otimes L \to L$ is called a Lie Algebra if $\forall a, b, c \in L$*

1. $[\ ,\ ] \circ T = -[\ ,\ ]$; *i.e.* $[a, b] = -[b, a]$ *(anti-symmetry)*
2. $[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$ *(Jacobi - identity)*

Notes:

1. Lie Algebras are non-associative
2. Due to (1), all ideals in a Lie Algebra L are two sided.
3. $[L, L]$ is always an ideal of L. If $[L, L] = (0)$, we call L an abelian Lie Algebra.

Definition 0.8. ***Group Variety*** *A group variety consists of a variety Y together with a homomorphism $\mu : Y \times Y \to Y$, such that the ste of points within Y have a binary operation μ which obey the normal group axoims.*

0.1 Cohomology in Algebraic Geometry

For any scheme X and any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules we want to define the groups $H^i(X,\mathcal{F})$. We can define by simply constructing $H^i(X,\mathcal{F})$ by firstly regarding X as a topological space (for our purposes this is the complex torus $X = V/\Lambda)$) and $\mathcal{F}$ as a sheaf of abelian groups. Letting $Ab(X)$ be the category of sheaves of abelian groups on X. We let $\Gamma = \Gamma(X,\cdot)$ be the global section functor from $Ab(X)$ in Ab, where Ab is the category of abelian groups. Recall that Γ is left exact so if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}\text{''} \to 0 \tag{0.3}$$

is an exact sequence in $Ab(X)$ then the following sequence is exact

$$0 \to \Gamma(\mathcal{F}') \to \Gamma(\mathcal{F}) \to (\mathcal{F}\text{''}) \tag{0.4}$$

in Ab.